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GENERALIZED PROBLEM OF THE IMPULSE PURSUIT OF A POINT WITH BOUNDED THRUST

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We consider a game problem [1, 2] similar in formulation to the problems in [3 – 5] and being a direct continuation of the results in [6]. Two material points of unit mass (the first and second players) move in a three-dimensional space under the action of controls F_1, F_2 alone. The control $u = F_1$ is bounded in total momentum, while the control $-v = F_2$ is bounded in absolute value. The game termination set M is an arbitrary fixed point in the space of relative positions and velocities of the players, while the payoff is the time taken to lead a relative trajectory to this point. The first player minimizes this time and the second maximizes it. The solution is in many respects analogous to the solution in [6] wherein the minimax time up to "hard" (with respect to the coordinates) and "soft" (with respect to the coordinates and velocities) contact of the points was determined. In the conclusion we consider the problem of soft contact of two controlled points in a linear position central gravity field. In the course of solving the problem in the title we form a vector-valued function $q(w, p)$ depending upon the game's position w and on a parameter p, and we divide the whole space W of possible positions into the regions W° and W_0. In region W° there exists a function $p_2(w) < 0$, defined as the smallest root of the equation $q(w, p) = 0$.

The first and second players' optimal controls are formed and the minimax time is computed in region W°. In the region W_0, where the equation $q(w, p) = 0$ either is entirely free of roots or admits of only nonnegative roots, the second player's control is formed, permitting him to evade falling onto set M under any action by the first player.

1. Let vectors r_1, r_2 define the positions of points m_1, m_2 relative to some fixed coordinate system, and let two fixed vectors b and a define the game termination set by the equations $M^\circ [r_1 - r_2 = b,\ r_1^{\cdot} - r_2^{\cdot} = a]$. Setting $x = r_1 - r_2 - b$, $y = r_1^{\cdot} - r_2^{\cdot}$, we compose the equations of relative motion in the form

$$x^{\cdot} = y, \qquad y^{\cdot} = u + v, \qquad \mu^{\cdot} = -|u| \tag{1.1}$$

$$\mu \geqslant 0, \qquad |v| \leqslant \nu \tag{1.2}$$

The last equation of system (1.1) in combination with the constraint $\mu \geqslant 0$ is equivalent to the "impulse" constraint

$$\mu_0 - \int_0^\tau |u|\, dt = \mu^{(1)}(\tau) \geqslant 0 \tag{1.3}$$

on the first player's control u. This constraint allows for jumps in the variables y, μ by the formulas

$$y^{(1)} = y + \mu_1, \qquad \mu^{(1)} = \mu - |\mu_1| \geqslant 0 \tag{1.4}$$

The vector w, defined by a collection of vectors and numbers $w = [x, y, a, \mu]$, is called a position, but a separate notation is allotted to the result $w^{(1)} = [x, y^{(1)} = y + \mu_1(w), a, \mu^{(1)} = \mu - |\mu_1|(w)]$ of the first player's impulse actions. Suppose that the vector $w^{(1)}$ $(t \geqslant 0)$ is specified as a function of time. Its initial value $w(0)$ is called the position at $t = 0$, while the left limit $w(\tau - 0)$ of vector $w^{(1)}$ is called the position $w(\tau > 0)$.

A pair of controls $u(w, v)$, $v(w)$ and the unique trajectory $w^{(1)}(t \geqslant 0, \{u(w, v), v(w)\}, w(0))$ corresponding to them are said to be admissible if the trajectory satisfies Eqs. (1.1) for almost all t, is right-continuous and satisfies constraints (1.2) for all t, admits of a finite number of jumps in accordance with formulas (1.4) on every finite interval $0 \leqslant t \leqslant t_1$, and is absolutely continuous on the intervals of continuity.

We project the vector y onto vector x and onto a plane normal to x. We obtain a projection y_α and a vector y_β and we introduce a right-hand triplet of unit basis vectors $j_\alpha, j_\beta, j_\gamma$ by the formulas $j_\alpha = x / |x|$ and $j_\beta = y_\beta / |y_\beta|$ for $|x| > 0$ and $|y_\beta| > 0$; $j_\alpha = x / |x|$, j_β, j_γ are arbitrary for $|x| > 0$, $|y_\beta| = 0$. Denoting the projections of vectors onto the unit vectors by subscripts α, β, γ we obtain the corollaries of Eqs. (1.1) in the form

$$|x|^{\cdot} = y_\alpha, \quad y_\alpha^{\cdot} = u_\alpha + v_\alpha + y_\beta^2 / |x| \tag{1.5}$$

$$|y_\beta|^{\cdot} = u_\beta + v_\beta - y_\alpha |y_\beta| / |x|, \quad a_\alpha^{\cdot} = a_\beta |y_\beta| / |x|$$

$$a_\beta^{\cdot} = -a_\alpha |y_\beta| / |x| \qquad \text{for } |x| > 0,\ |y_\beta| > 0.$$

Equations (1.5) are preserved when $|x| > 0$, $|y_\beta| = 0$, excepting the equation for $|y_\beta|^{\cdot}$ which acquires the form

$$|y_\beta|^{\cdot} = [(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2]^{1/2}$$

We write the jump Eqs. (1.4) in the form

$$y_\alpha^{(1)} = y_\alpha + \mu_{1\alpha}, \quad |y_\beta^{(1)}| = [(|y_\beta| + \mu_{1\beta})^2 + \mu_{1\gamma}^2]^{1/2}$$

Since it is intuitively clear that the problem's solution depends only on the collection of quantities $|x|$, y_α, y_β, a_α, a_β, μ, we retain the notation w for it, and only when $|x| = 0$, we interpret the quantity w as the collection of y, a, μ.

When $\nu > 0$ we can obtain $\nu = 1$ by norming the variables. Therefore, in what follows we examine two cases: $\nu = 1$ and $\nu = 0$. The possibility of step variations in velocity y transforms set M° into the set

$$M\,[\,|x| = 0, \quad \mu \geqslant |y - a|\,]$$

The approach to the set $|x| = 0$ at instant $\tau > 0$ is accompanied by the condition $y_\beta(\tau) = 0$; therefore, to settle the question of belonging to set M it is sufficient to know the sign of the difference

$$\mu - [a^2 - 2a_\alpha(\tau)y_\alpha(\tau) + y_\alpha^2(\tau)]^{1/2}$$

However, if $|x(0)| = 0$ at the initial instant, then to settle the question of belonging to M we need to know the sign of the difference $\mu - |y - a|$, therefore, we consider the vectors a, y as known when $|x| = 0$.

2. In analogy with [6] we assume that there exists a certain function $p_0(w) < 0$ which satisfies the estimate

$$q(w, p_0) = \mu - l_1(w, p_0) - l_2(w, p_0) + |x| / p_0 \geqslant 0 \qquad (2.1)$$

Here

$$l_1(w, p) = \sqrt{a_\theta^2 + (p - a_\alpha)^2}, \qquad l_2(w, p) = \sqrt{y_\beta^2 + (p - y_\alpha^2)}$$

$$a_\theta = a_\beta j_\beta + a_\gamma j_\gamma$$

Furthermore, let the function $p_0(w)$ be such that from the equality

$$p_0(|x|, y_\alpha, |y_\beta| = 0, a_\alpha, a_\beta, \mu) - y_\alpha = 0 \qquad (2.2)$$

follows the equality

$$p_0(|x_1|, y_\alpha, |y_\beta| = 0, a_\alpha, a_\beta, \mu^1) - y_\alpha = 0 \qquad (2.3)$$

for any $|x_1| \leqslant |x|$, $\mu^1 \leqslant \mu$. Then the control

$$u_1(w, v) = +(p_0 - y_\alpha)\delta j_\alpha - |y_\beta|\,\delta j_\beta, \qquad w \in [l_2(w, p_0) > 0]$$

$$u_1(w, v) = -v, \qquad w \in [l_2(w, p_0) = 0]$$

brings the trajectory onto set M in time

$$T_1(w) = -|x| / |p_0|(w)$$

In fact, at the initial instant we have the equality

$$w^{(1)} = [|x|, p_0(w), |y_\beta^{(1)}| = 0, \quad a_\alpha, a_\beta, \mu^{(1)} = \mu - l_2(w, p_0)]$$

Subsequently a rectilinear uniform motion takes place in accordance with the equations

$$|x|^\cdot = p_0(w), \qquad y_\alpha^\cdot = y_\beta^\cdot = 0, \qquad \mu^\cdot = -|v|$$

and the control $u_1(w, v) = -v$ is realized along this motion according to the constraints (2.2), (2.3).

The question of the possibility of realizing the control $u_1(w, v)$ leads naturally to the problem of investigating the maximum of the function $q(w, p)$ in the region $p < 0$. We obtain the expressions

$$q^{(1)} = (a_\alpha - p) / l_1 + (y_\alpha - p) / l_2 - |x| / p^2$$
$$q^{(2)} = -a_\theta^2 / l_1^3 - y_\beta^3 / l_2^3 + 2|x| / p^3 < 0$$

for the first, $q^{(1)}$, and the second, $q^{(2)}$, partial derivatives of the function $q(w, p)$ with respect to the variable p. These formulas show that when $w \in D_1 [|x| > 0, |a_\theta| > 0, |y_\beta| > 0]$ there exists a unique continuously-differentiable function $p_1(w) < 0$ corresponding to the equalities

$$q^{(1)}(w, p_1(w)) = 0, \qquad r(w) = q(w, p_1(w)) = \max_{p<0} q(w, p)$$

The regions

$$D_2 [|x| > 0, |a_\theta| = 0, |y_\beta| > 0]$$
$$D_3 [|x| > 0, |a_\theta| > 0, |y_\beta| = 0]$$
$$D_4 [|x| > 0, |a_\theta| = |y_\beta| = 0]$$

need a more detailed investigation. In region D_2 we have the equalities

$$q^{(1)}(w, p - a_\alpha < 0) = 1 + (y_\alpha - p) / l_2 - |x| / p^2$$
$$q^{(2)}(w, p - a_\alpha > 0) = -1 + (y_\alpha - p) / l_2 - |x| / p^2 < 0$$

The following alternative is a corollary of these relations.

The point of maximum of $p_1(w)$ corresponds to the equation

$$q^{(1)}(w, p - a_\alpha < 0) = 1 - (y_\alpha - p) / l_2(w, p) - |x| / p^2 = 0 \quad (2.4)$$

if the position

$$w \in D_{2,1} = D_2 \cap \{[a_\alpha \geqslant 0] \cup [a_\alpha < 0, \ q_2(w) = 1 + (y_\alpha - a_\alpha) / l_2(w, a_\alpha) - x / a_\alpha^2 < 0]\}$$

The point of maximum of $p_1(w)$ corresponds to the equation

$$p_1(w) = a_\alpha \quad (2.5)$$

if the position

$$w \in D_{2,2} = D_2 \cap [a_\alpha < 0, \ q_2(w) \geqslant 0]$$

The investigation in regions D_3, D_4 is carried out analogously:

$$q^{(1)}(w, p_1 - y_\alpha < 0) = (a_\alpha - p_1) / l_1(w, p_1) + 1 - |x| / p_1^2 = 0 \quad (2.6)$$

if the position

$$w \in D_{3,1} = D_3 \cap \{[y_\alpha \geqslant 0] \cup [y_\alpha < 0, \ q_3(w) = (a_\alpha - y_\alpha) / l_1(w, y_\alpha) + 1 - |x| / y_\alpha^2 < 0]\}$$

The point of maximum of $p_1(w)$ corresponds to the equation

$$p_1(w) = y_\alpha \quad (2.7)$$

if the position

$$w \in D_{3,2} = D_3 \cap [y_\alpha < 0, q_3(w) \geqslant 0]$$

One of two cases is realized in region D_4:

$$p_1(w) = -\sqrt{|x|/2}, \quad w \in D_{4,1} = D_4 \cap [-\sqrt{|x|/2} < \min(y_\alpha, a_\alpha)] \tag{2.8}$$

$$p_1(w) = \min(y_\alpha, a_\alpha), \quad w \in D_{4,2} = D_4 \cap [-\sqrt{|x|/2} \geqslant \min(y_\alpha, a_\alpha)] \tag{2.9}$$

Thus, the function $p_1(w) < 0$ and the function $r(w) = \max_{p<0} q(w, p)$ are defined by the relations $q^{(1)}(w, p) = 0$, (2.4) – (2.9) in regions $D_1, \ldots, D_4$.

3. We start on the construction of the control $v_0(w)$ solving the problem

$$r^{\cdot}(w, u_0(w, v), v_0(w)) = \min_v \max_u r^{\cdot}(w, u(w, v), v(w))$$

In the region $D_5\ [|x| > 0,\ l_2(w, p_2(w)) > 0]$ this construction encounters no difficulties and leads to the equation

$$v_0(w) = (-(p_1(w) - y_\alpha) j_\alpha + |y_\beta| j_\beta) / l_2(w, p_1), \quad w \in D_5$$

In the regions

$$D_6\ [|x| > 0,\ l_2(w, p_1(w)) = 0,\ l_1(w, p_1(w)) > 0]$$

$$D_7\ [|x| > 0,\ l_2(w, p_1(w)) = 0,\ l_1(w, p_1(w)) = 0]$$

the computation of $r^{\cdot}(w, u, v)$ meets with difficulty. In these regions, by specifying the control u, v, we compute $p_1(w + \Delta w)$ by those formulas from the collection $q^{(1)}(w + \Delta w,\ p_1(w + \Delta w)) = 0$, (2.4) – (2.9), which correspond to the vector

$$w + \Delta w = w + w^{\cdot}(w, u, v)\Delta t$$

belonging to regions D_1, $D_{2,1}, \ldots, D_{4,2}$, respectively. In the majority of cases the passage to the limit as $\Delta t \to 0$ in these formulas makes it possible to compute $p_1^{\cdot}(w, u, v)$. In those cases when the derivative $p_1^{\cdot}(w, u, v)$ becomes infinite, it is possible to compute $p_1(w + \Delta w)$, and then to compute $r(w + \Delta w)$ and the derivative $r^{\cdot}(w, u, v)$ which proves to be finite.

Having applied the technique described, we arrive at the expression

$$v_0(w) = -s j_\alpha + \sqrt{1 - s^2}\, j_\beta$$

$$s = \begin{cases} 1 - |x| / a_\alpha^2, & w \in D_6 \\ (a_\alpha - y_\alpha) / l_1(w, y_\alpha) - |x| / y_\alpha^2, & w \in D_7 \end{cases}$$

We can show that when $|x| > 0$ the function $s(w)$ is contained within the limits $1 > s(w) \geqslant -1$ and, therefore, it is always possible to take the square root.

We extend the functions $r(w)$ and $v_0(w)$ onto the set $M_1\ [|x| = 0,\ \mu - |a - y| < 0]$ by the formulas

$$r(w) = \mu - |a - y|, \qquad v_0(w) = y - a / |y - a|$$

and we go on to prove a lemma.

Lemma 3.1. The function $r(w)$ does not increase along any trajectory determined by an admissible pair $u(w, v)$, $v_0(w)$.

The proof of Lemma 3.1 is in many respects analogous to the proof of the corresponding lemma in [6] and, therefore, we shall not prove certain statements, restricting ourselves to making an appropriate reference.

3.1.1. Any impulse control $u = \mu_1\delta$ does not increase function $r(w)$ [6].

3.1.2. The right derivative $r^{\cdot}(w, u, v_0(w))$ is nonpositive; moreover,

$$r^{\cdot}(w, u, v_0(w)) < 0 \qquad w \in D_5 \tag{3.1}$$

$$r^{\cdot}(w, u \neq u_0(w, v), v_0(w)) < 0, \qquad w \in (D_6 \cup D_7) \cap [s > -1] \tag{3.2}$$

$$r^{\cdot}(w, u_0(w, v), v_0(w)) = 0 \qquad w \in D_6 \cup D_7 \tag{3.3}$$

To prove estimate (3.3) we compute the derivative $r^{\cdot}$ separating it into two terms

$$r^{\cdot}(w, u, v_0(w)) = R_1(w) + R_2(w, u)$$

$$R_1(w) = y_\alpha / p_1 + p_1\alpha_\beta \mid y_\beta \mid / \mid x \mid l_1 + p_1 y_\beta^2 / \mid x \mid l_2 - 1$$

$$R_2(w, u) = - \mid u \mid + (p_1 - y_\alpha)u_\alpha / l_2 - \mid y_\beta \mid u_\beta / l_2$$

$$p_1 = p_1(w), \quad l_1 = l_1(w, p_1), \quad l_2 = l_2(w, p_1)$$

By simple manipulations we obtain the equality

$$R_1(w) = p_1 / \mid x \mid [(\mid x \mid / p_1^2)(y_\alpha - p_1) + a_\beta \mid y_\beta \mid / l_1 + y_\beta^2 / l_2]$$

Having replaced in the first term within the brackets the factor $\mid x \mid / p_1^2$ by the sum $((a_\alpha - p_1) / l_1 + (y_\alpha - p_1) / l_2)$, from the equation $q^{(1)}(w, p_1(w)) = 0$ we obtain

$$R_1(w) = p_1 [l_1 l_2 + (y_\alpha - p_1)(a_\alpha - p_1) + a_\beta \mid y_\beta \mid] / l_1 \mid x \mid < 0 \tag{3.4}$$

The last estimate is a consequence of the relations

$$p_1(w) < 0, \quad (y_\alpha - p_1) / l_2 + (a_\alpha - p_1) / l_1 = \mid x \mid / p_1^2 \tag{3.5}$$

In fact, the equating to zero of the left-hand side of the second relation in (3.5) is a consequence of the assumption that the expression within brackets in (3.4) equals zero.

Note. Estimate (3.4) together with the estimate $R_2(w, u) \leqslant 0$ ascertains estimate (3.1) when $w \in D_5 \cap [l_1 > 0, l_2 > 0]$. In the region $w \in D_5 \cap [l_1 = 0, l_2 > 0]$ estimate (3.1) is ascertained analogously. Relations (3.2) and (3.3) were established in region $D_{4,2} \cap [a_\alpha = y_\alpha < 0, -\sqrt{\mid x \mid /2} < a_\alpha]$ when constructing $v_0(w)$. These relations can be verified analogously in the remaining parts of the regions $(D_6 \cup D_7) \cap [s > -1]$, $D_6 \cup D_7$.

3.1.3. Region $(D_6 \cup D_7) \cap [s = -1]$ is characterized by the test that any control u preserving the estimate $r^{\cdot}(w, u, v_0(w)) \geqslant 0$ realizes the equality $r^{\cdot}(w, u, v_0(w)) = - \mid u \mid - u_\alpha = 0$ for $u_\alpha < 0$, $u_\beta = u_\gamma = 0$. However, it can be shown that this control cannot be an impulse control because any impulse control $u = -\mu_1\delta j_\alpha$ diminishes function $r(w)$.

3.1.4. The increment $\Delta r(w, u, \Delta t, v_0(w)\Delta t) \leqslant 0$ for $w \in M_1$ and for small Δt.

The proof is carried out by direct verification. This completes the proof of Lemma 3.1. The next theorem is a corollary of Lemma 3.1.

Theorem 3.1. If $w(0) \in W_0 = [\mid x \mid > 0, r(w) < 0] \cup M_1$, then there is no admissible pair $u(w, v)$, $v_0(w)$ which can bring the trajectory onto set M in finite time. The proof is analogous to the proof of the corresponding theorem in [6](*).

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*) The continuity of the functions $p_1(w)$, $r(w)$, $p_2(w)$, $T^0(w)$ is estsblished analogously as in [6].

4. A function $p_2(w)$, the smallest root of the equation $q(w, p) = 0$, exists in the region $D^\circ\ [|x| > 0,\ r(w) \geqslant 0]$.

Theorem 4.1. The pair of controls

$$\left.\begin{aligned} u^\circ(w) &= (p_2 - y_\alpha)\delta j_\alpha - |y_\beta|\delta j_\beta \\ v^\circ(w) &= -u^\circ(w)/|u^\circ(w)| \end{aligned}\right\} \quad w \in D^\circ \cap [l_2(w, p_2) > 0] \tag{4.1}$$

$$\left.\begin{aligned} u^\circ(w, v) &= -v \\ v^\circ(w) &= v_0(w) \end{aligned}\right\} \quad w \in D^\circ \cap [l_2(w, p_2) = 0] \tag{4.2}$$

realize a time

$$T[u^\circ, v^\circ] = T^\circ(w) = -|x| / p_2(w).$$

of hitting onto the set M, and this time satisfies the estimate

$$T[u, v^\circ] \leqslant T[u^\circ, v^\circ] \leqslant T[u^\circ, v]$$

for any admissible pairs $(u(w, v), v^\circ(w))$, $(u^\circ(w, v), v(w))$.

The proof of Theorem 4.1 consists in proving a number of assertions listed below.

4.1.1. Any impulse control $u = \mu_1\delta \neq mu^\circ(w)$ $(0 \leqslant m \leqslant 1)$ either strictly diminishes the function $T^\circ(w)$ or transfers the position into region W_0 [6].

4.1.2. Any finite control $u(w, v)$ with $w \in [r(w) > 0,\ l_2(w, p_2) > 0]$ realizes the estimate $T^{\circ\cdot}(w, u, v^\circ(w)) > -1$.

Proof. For $w \in [r(w) > 0,\ l_2(w, p_2) > 0,\ l_1(w, p_2) > 0]$ the derivative $T^{\circ\cdot}$ can be obtained in the form

$$T^{\circ\cdot}(w, u, v^\circ(w)) = -1 + T_1(w) + T_2(w, u) \quad T_1(w) =$$
$$-[l_1(w, p_2)\, l_2(w, p_2) + (a_\alpha - p_2)(y_\beta - p_2) + a_\beta |y_\beta| / p_2 q^{(1)} l_1(w, p_2);$$
$$q^{(1)} = q^{(1)}(w, p_2) = (a_\alpha - p_2)/l_1(w, p_2) + (y_\alpha - p_2)/l_2(w, p_2) - |x| / p_2^2 > 0$$
$$T_2(w, u) = -[-|u| + (p_2 - y_\alpha)u_\alpha / l_2(w, p_2) - |y_\beta| u_\beta / l_2(w, p_2)] / p_2 q^{(1)}$$

Arguments analogous to those applied in the proof of assertion 3.1.2 of Lemma 3.1 allow us to obtain the estimate $T_1(w) > 0$ on the basis of the estimates $p_2 < 0$, $q^{(1)}(w, p_2) > 0$. We recall that the estimate $q^{(1)}(w, p_2) > 0$ is a consequence of the definition of $p_2(w)$ as the smallest root of the equation $q(w, p) = 0$. The estimates $T_1(w) > 0$, $T_2(w, u) \geqslant 0$ complete the proof of assertion 4.1.2 for $l_1(w, p_2) > 0$. The case $l_1(w, p_2) = 0$ is proved analogously.

4.1.3. In region

$$[r(w) = 0,\ l_2(w, p_2) > 0] \cup [r(w) = l_2(w, p_2) = 0,\ s > -1]$$

any control $u \neq u^\circ(w)$ $(u \neq u^\circ(w, v) = -v)$ in pair with $v^\circ(w)$ transfers the position into region W_0.

To prove this assertion it suffices to note that $p_1(w) = p_2(w)$ in the region indicated and to remember that any control $u \neq u^\circ(w)$ with $w \in [r(w) = 0,\ l_2(w, p_1) > 0]$ realizes the estimate $r^\cdot(w, u, v^\circ(w) = v_0(w)) < 0$, while any control $u \neq -v$ with $w \in D^\circ \cap [r(w) = l_2(w, p_2) = 0,\ s > -1]$ realizes either the estimate $r^\cdot(w, u, v_0(w)) < 0$ or the equality $r^\cdot(w, u, v_0(w)) = 0$, but here leads the position into the region $D^\circ \cap [r(w) = 0,\ l_2(w, p_1) > 0]$.

4.1.4. As was noted in paragraph 3.1.3, in the region $[r(w) = l_2(w, p_2) = 0,\ s = -1]$ the equality $r^\cdot(w, u, v_0(w)) = 0$ is preserved by any control $u = u_\alpha j_\alpha$,

$u_\alpha < 0$, under the condition that u_α is sufficiently large in absolute value, i.e. $s^\circ(w, u, v_0) \geqslant 0$. However, the realization of a control $u_\alpha < 0$, being sufficiently large in absolute value but not having the nature of an impulse, cannot change the time $T^\circ(w)$, since at the next instant t the position hits onto the set

$$[r(w) = l_2(w, p_2) = 0,\ s > -1]$$

Assertions 4.1.1 – 4.1.4 settle the proof of the first estimate in system (4.2). To prove the second estimate it is sufficient to determine the easily-verifiable equality $T[u^\circ, v^\circ] = T[u^\circ, v]$. This completes the proof of Theorem 4.1.

5. Let $\nu = 0$; then we arrive at the time-optimal problem for set M. In this case many of the calculations can be carried out in explicit form. However, certain difficulties arise. The first difficulty is that the function $q(w, p) = \mu - l_1(w, p) - l_2(w, p)$ may not admit of a stationary maximum point in the region $p \leqslant 0$, while a second one is encountered when the stationary points fill up an entire segment $a_\alpha \leqslant p \leqslant y_\alpha$. Denoting by $p_1(w)$ the smallest of all p for which the function $q(w, p)$ reaches a maximum, and by $p_2(w)$, as before, the smallest root of equation $q(w, p) = 0$, we arrive at the results of the investigation

$$\begin{aligned}
&p_1(w) = (a_\alpha |y_\beta| + |a_\theta| y_\alpha) / (|y_\beta| + |a_\theta|) \\
&r(w) = \mu - \sqrt{(|y_\beta| + |a_\theta|)^2 + (a_\alpha - y_\alpha)^2} \\
&|a_\theta| = \sqrt{a_\beta^2 + a_\gamma^2} \\
&w \in [D_1 \cup D_2 \cup D_3] \cap [a_\alpha |y_\beta| + |a_\theta| y_\alpha \leqslant 0]
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&p_1(w) = \min(a_\alpha, y_\alpha), \quad r(w) = \mu - |a_\alpha - y_\alpha| \\
&w \in D_4 \cap [\min(a_\alpha, y_\alpha) \leqslant 0]
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&p_1(w) = 0, \quad r(w) = \mu - |a| - |y| \\
&w \in \{[D_1 \cup D_2 \cup D_3] \cap [a_\alpha |y_\beta| + |a_\theta| y_\alpha > 0]\} \cup \\
&\quad \{D_4 \cap [\min(a_\alpha, y_\alpha) > 0]\}
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
&p_2(w) = \lambda_2^{-1}(\lambda_3 - \sqrt{\lambda_3^2 - \lambda_2\lambda_4}) \\
&\lambda_2 = 2\lambda_5(a_\alpha - y_\alpha) = 2a_\alpha\mu^2, \qquad 2\lambda_5 = \mu^2 + a^2 - y^2 \\
&\lambda_3 = \mu^2 - (a_\alpha - y_\alpha)^2, \qquad \lambda_4 = \lambda_5^2 - \mu^2 a^2 \\
&w \in [r(w) > 0]
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&p_2(w) = \min(a_\alpha, y_\alpha) \\
&w \in D_4 \cap [r(w) = 0, \min(a_\alpha, y_\alpha) < 0]
\end{aligned} \tag{5.5}$$

The control $u^\circ(w)$ is formed by the same rules as above. However, it should be noted that in the set W_0 from which it is impossible to hit onto set M we have to include, besides the set $[r(w) < 0]$, also the set $D_0^1\,[r(w) = 0,\ p_1(w) = 0]$. The proof of the last statement is based on the fact that the set $D_0^1\,[r(w) = p_1(w) = 0]$ does not contain set M, and any control u which shifts the position from set D_0^1 necessarily transfers it into the set $[r(w) < 0]$.

6. The geometric interpretation of the optimal motion is as follows. Suppose that at the initial instant the vectors x, y, a lie in one plane (see Fig. 1). The first player's optimal action is the impulse $u^\circ(w)$ transfering the position w into the vector

$$w^1\ [|x|,\ y_\alpha^{\ 1} = p_2(w),\ y_\beta^{(1)} = 0,\ a_\alpha,\ a_\beta,\ \mu^{(1)} = \mu - |u^\circ(w)|].$$

For $t > 0$ the second player realizes the control $v_0(w)$ with component $v_\theta = v_\beta i_\beta + v_\gamma i_\gamma$ directed arbitrarily in the plane, perpendicular to vector x, and with modulus $|v_\theta| = \sqrt{v_\beta^2 + v_\gamma^2}$, which follows a circle of unit radius (Fig. 2). The center of this circle is located on the $|x| / y_\alpha^{\ 2}$ axis at the point

$$|x| / y^2_\alpha = (a_\alpha - y_\alpha) / l_1(w, y_\alpha)$$

The component v_α follows the straight line $v_\alpha = -(a_\alpha - y_\alpha) / l_1(w, y_\alpha) + |x| / y_\alpha^{\ 2}$.

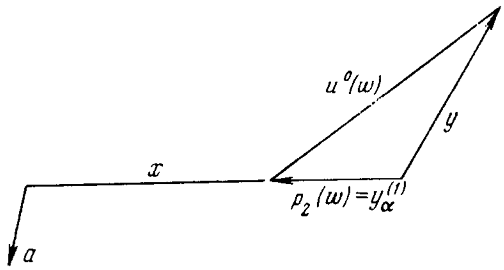

Fig. 1

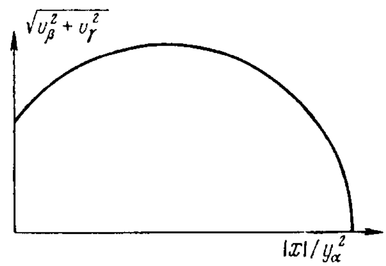

Fig. 2

For $t > 0$ the first player, using the control $u^\circ(w, v) = -v$ obstructs his opponent's action and the motion takes place along the straight line $x = x(0)$ with constant velocity $y_\alpha = p_2(w)$ in the region $[l_2(w, p_2) = 0 = r(w)]$.

7. We now assume that the forces $f_{1,2} = -\omega^2 r_{1,2}$ of attraction to a fixed center o act on the points in addition to the controls, while the set $M^\circ\ [x = r_1 - r_2 = y = r_1^{\cdot} - r_2^{\cdot} = 0]$ corresponds to a "soft" contact with respect to the coordinates and velocities. After a suitable norming we obtain $\omega^2 = 1$, and the equations of relative motion, the set M, and the function $q(w, p)$ take the form

$$|x|^{\cdot} = y_\alpha^{\cdot}, \quad y_\alpha = -|x| + y_\beta^{\ 2} / |x| + u_\alpha + v_\alpha$$
$$|y_\beta|^{\cdot} = -y_\alpha |y_\beta| / |x| + u_\beta + v_\beta$$
$$M\ [x = 0,\ \mu - |y| \geqslant 0]$$
$$q(w, p) = \mu - \sqrt{p^2 + x^2} - \sqrt{y_\beta^{\ 2} + (p - y_\alpha)^2} + \operatorname{arctg}(p / |x|) + \pi / 2$$

In contrast to the preceding, we shall subsequently denote the quantity $\sqrt{p^2 + x^2}$ by $l_1(w, p)$ and, independently of the preceding, we shall also designate regions $D_{i,j}$ of the phase space. The remaining notation is retained from the preceding.

Computing $q^{(1)}$ and $q^{(2)}$, we obtain

$$q^{(1)} = -p / l_1(w, p) - (p - y_\alpha) / l_2(w, p) - |x| / l_1^2(w, p) \qquad (7.1)$$
$$q^{(2)} = -|x|^2 / l_1^3(w, p) - y_\beta^{\ 2} / l_2^3(w, p) + 2p\,|x| / l_1^4(w, p) \quad (7.2)$$
$$l_1(w, p) = \sqrt{p^2 + x^2}, \qquad l_2(w, p) = \sqrt{y_\beta^{\ 2} + (p - y_\alpha)^2}$$

From $q^{(1)} = 0$ follows the equation

$$2p_1 \mid x \mid / l_1^4 = -2p_1^2 / l_1^3 - 2p_1 (p_1 - y_\alpha) / l_2 l_1^2$$

therefore, at the stationary point $p = p_1 (w)$ of the function $q (w, p)$ the second derivative $q^{(2)}$ has the form

$$q^2 (w, p_1 (w)) = l_1^{-1} [-y_\beta^2 / l_2^2 - l_1 y_\beta^2 / l_2^3 - (p_1 / l_1 + (p_1 - y_\alpha) / l_2)^2] < 0$$

This estimate establishes that for $w \in D_1$ there exists a unique continuously-differentiable function $p_1 (w)$ corresponding to the equation $q^{(1)} (w, p) = 0$ and to the equality

$$r (w) = \max_p q (w, p) = q (w, p_1 (w))$$

The derivative $q^{(1)} (w, p)$ is discontinuous in the region D_2 $[\mid x \mid > 0, \mid y_\beta \mid = 0]$: therefore, the investigation is complicated somewhat. Let us cite its result.

Either $p_1 (w)$ is determined from the equation

$$q^{(1)} (w, p - y_\alpha < 0) = -p / l_1 (w, p) + 1 - \mid x \mid / l_1^2 (w, p) = 0 \quad (7.3)$$

for

$$w \in D_{2,1} = D_2 \cap [q_2 (w) = -y_\alpha / l_1 (w, y_\alpha) + 1 - \mid x \mid / l_1^2 (w, y_\alpha) < 0]$$

or

$$p_1 (w) = y_\alpha, \quad w \in D_{2,2} = D_2 \cap [q_2 (w) \geqslant 0] \quad (7.4)$$

After substituting $p_1 (w)$ into function $q (w, p)$ we obtain the function $r (w)$ for $w \in D_2$. This function proves to be continuously differentiable for $w \in D_1 \cup D_{2,1}$ and admits of discontinuities in the partial derivatives for $w \in D_{2,2}$. By analogy with [6] it can be shown that the functions $p_1 (w)$ and $r (w)$ themselves remain continuous.

Let us assume that at a given position w both players use finite controls $u (w, v)$, $v (w)$. We compute the right derivative $r^{\cdot} (w, u (w, v), v (w))$, and then we determine $v_0 (w)$ corresponding to the equality

$$\min_v \max_u r^{\cdot} (w, u (w, v), v (w)) = r^{\cdot} (w, u_0 (w, v), v_0 (w))$$

As a result of this construction (it encounters certain difficulties in the region $w \in D_{2,2}$) we obtain

$$v_0 (w) = (- (p_1 (w) - y_\alpha) j_\alpha + \mid y_\beta \mid j_\beta) / l_2 (w, p_1 (w)), \quad (7.5)$$
$$w \in D_1 \cup D_{2,1}$$

$$v_0 (w) = -s (w) j_\alpha + \sqrt{1 - s^2 (w)} j_\beta, \quad w \in D_{2,2} \quad (7.6)$$
$$s (w) = -y_\alpha / l_1 (w, y_\alpha) - \mid x \mid / l_1^2 (w, y_\alpha)$$

We can show that when $w \in D_{2,2}$ the function $s (w)$ lies within the limits $1 > s (w) \geqslant -1$ and, therefore, it is always possible to take the root $\sqrt{1 - s^2 (w)}$. We note also that since the direction of vector j_β in the plane, normal to vector x, can be taken arbitrarily when $w \in D_{2,2}$, the control $v_0 (w)$ has the same arbitrariness.

We extend the functions $r (w)$, $v_0 (w)$ onto the set M_1 $[\mid x \mid = 0, \mu - \mid y \mid < 0]$ by the formulas $r (w) = \mu - \mid y \mid$, $v_0 (w) = y / \mid y \mid$ and we prove a lemma.

Lemma 7.1. The function $r (w)$ is continuous in t at the points of continuity of an admissible trajectory and does not increase along the trajectory corresponding to any admissible pair $u (w, v)$, $v_0 (w)$.

7.1.1. The continuity of the functions $r(w)$, $p_1(w)$ in the region $|x| > 0$ is proved in the same way as in [6]. Let $w_i(t_i)$ be a sequence of points of an admissible trajectory, converging as $i \to \infty$ to a position $w(\tau) \in M_1$. Then the relations

$$y_{\beta i}(t_i) \to 0, \quad y_{\alpha i}(t_i) \to \pm |y(\tau)| \quad \text{as} \quad i \to \infty$$

are obvious; the minus sign corresponds to the case $t_{i+1} > t_i$ and the plus to the case $t_{i+1} < t_i$. In both cases formulas (7.1), (7.3), (7.4) yield the equality

$$\lim (|x(t_i)| / l_1(w(t_i), p_1(w(t_i)))) = 0$$

and show as well that for sufficiently large i the quantities

$$p_1(w(t_i)), \qquad p_1(w(t_i)) - y_{\alpha i}(t_i)$$

cannot have like signs; two versions are possible

$$0 \leqslant p_1(w(t_i)) \leqslant y_{\alpha i}(t_i) > 0$$
$$0 \geqslant p_1(w(t_i)) \geqslant y_{\alpha i}(t_i) < 0$$

The latter relations establish the equality

$$\lim r(w_i(t_i)) = \mu(\tau) - |y(\tau)|$$

and complete the proof of the first assertion of the lemma.

7.1.2. Any impulse control $u = \mu_1\delta$ does not increase the function $r(w)$ [6].

7.1.3. The right derivative $r^{\cdot}(w, u(w, v), v_0(w)) \leqslant 0$ for any $w \in [|x| > 0]$ and at any control $u(w, v)$.

We prove this assertion for $w \in D_1 \cap D_{2,1}$. Computing the derivative $r^{\cdot}(w, u, v_0(w))$, we represent it as a sum of two terms

$$r^{\cdot}(w, u, v_0(w)) = R_1(w) + R_2(w, u) + R_3(w, v_0)$$
$$R_3 + R_1(w) = -|x| y_\alpha / l(w, p_1) + (|x| y_\alpha + p(-|x| + y_\beta^2 / |x|)) / l_2(w, p_1) + p_1 y_\alpha / l_2^2(w, p_1) - 1$$
$$R_2(w, u) = -|u| - (p_1 - y_\alpha) u_\alpha / l_2(w, p_1) - |y_\beta| u_\beta / l_2(w, p_1)$$
$$p_1 = p_1(w)$$

Adding the expression $p_1 |x| / l_1(w, p_1) - p_1^2 / l_1^2(w, p_1)$ to the quantity $R_1(w)$ and subtracting this same expression, we give the expression for $R_1 + R_3$ the form

$$R_1(w) + R_3(w, v_0) = |x|^{-1} \{(p_1 - y_\alpha) |x|^2 / l_1 - p_1(p_1 - y_\alpha) |x| / l_1^2 + p_1 y_\beta^2 / l_2\} + |x| [-p_1 / l_1 - (p_1 - y_\alpha) / l_2 - |x| / l_1^2]$$
$$l_1 = l_1(w, p_1), \quad l_2 = l_2(w, p_1)$$

The brackets vanish by virtue of the equation $q^{(1)}(w, p_1) = 0$. Replacing the factor of the second term $|x| / l_1^2$, appearing in braces, by the sum $-p_1 / l_1 - (p_1 - y_\alpha) / l_2$ from the equation $q^{(1)}(w, p_1) = 0$ and making some elementary manipulations, we obtain

$$R_1(w) + R_3(w, v_0) = |x|^{-1} [(p_1 - y_\alpha) l_1 + p_1 l_2] = -l_2(w, p_1) / l_1(w, p_1) < 0$$

The last estimate, together with the trivial estimate $R_2(w, u) \leqslant 0$, establishes the estimate $r^{\cdot}(w, u, (w, v), v_0(w)) < 0$ for $w \in D_1 \cup D_{2,1}$. The estimate $r^{\cdot}(w, u(w, v), v_0$

$(w)) \leqslant 0$ for $w \in D_{2,2}$ follows from the equality

$$r^{\cdot} = 1 - |u| + s(u_\alpha + v_\alpha) - \sqrt{1-s^2}\sqrt{(u_\beta + v_\beta{}^2) + (u_\gamma + v_\gamma)^2}$$

whose proof is cumbersome and is not carried out here. We merely note one of its properties: when $w(0) \in D_{2,2}$ there exists only one trajectory $w_1^{(1)}$ $(t \geqslant 0,$ $\{u^\circ(w, v) = -v, v_0(w)\}, w(0))$, along which the equality $r(w_1(t)) = r(w(0))$ is preserved, while any control $u(w, v)$ which in pair with $v^\circ(w)$ shifts the position from the trajectory $w_1{}^{(1)}(t)$ leads to diminishing $r(w)$. The diminishing of $r(w)$ under a shift of positions from set M_1 can be verified by direct calculation; this completes the proof of Lemma 7.1 and allows to state a theorem.

Theorem 7.1. If the initial position $w(0) \in W_0$ $[r(w) < 0]$, then any pair $u(w, v)$, $v_0(w)$ does not lead an admissible trajectory onto set M in finite time [6].

8. In the region W° $[r(w) \geqslant 0,$ $|x| > 0]$ we define the function $p_2(w)$ as the smallest root of the equation $q(w, p) = 0$ and we form the controls

$$u^\circ(w, v) = (p_2 - y_\alpha)\delta j_\alpha - |y_\beta|\delta j_\beta$$
$$v^\circ(w) = (-(p_2 - y_\alpha)j_\alpha + |y_\beta| j_\beta) / l_2(w, p_2)$$
$$p_2 = p_2(w), \qquad w \in D_1{}^\circ = W^\circ \cap [l_2(w, p_2) > 0]$$
$$u^\circ(w, v) = -v, \qquad v^\circ(w) = v_0(w)$$
$$w \in D_2{}^\circ = W^\circ \cap [l_2(w, p_2) = 0]$$

Let us clarify the structure of the region $D_2{}^\circ$. The condition $l_2(w, p_2) = 0$ implies the equalities $|y_\beta| = 0$, $p_2(w) = y_\alpha$. From the definition of $p_2(w)$ as the smallest root of the equation $q(w, p) = 0$ we have the estimate

$$\lim q^{(1)}(w, p - y_\alpha \rightarrow -0) = q_1(w) \geqslant 0$$

This signifies that the equality $y_\alpha = p_1(w) = p_2(w)$ holds according to (7.4). Thus, in another notation we can write $D_2{}^\circ = D_{2,2} \cap [r(w) = 0]$. The latter equality points up the possibility of forming $v_0(w)$ from formulas (7.6) for $w \in D_2{}^\circ \in D_{2,2}$.

Theorem 8.1. The pair $u^\circ(w, v)$, $v^\circ(w)$ realizes the time

$$T[u^\circ, v^\circ] = T^\circ(w) = \operatorname{arctg}(p_2(w) / |x|) + \pi / 2$$

and the time $T[u, v]$ of first hitting onto set M, corresponding to the admissible pair $u(w, v)$, $v(w)$, satisfies the estimates

$$T[u^\circ(w, v), v] \leqslant T[u^\circ, v^\circ] \leqslant T[u(w, v), v^\circ(w)]$$

The proof of Theorem 8.1 relies on the successive proofs of the following assertions.

8.1.1. Any impulse control $u = \mu_1\delta$ does not diminish the function $p_2(w)$ (the function $T^\circ(w)$), i.e. the estimate

$$\Delta p = p_2(w^{(1)}) - p_2(w) \geqslant 0 \qquad (\Delta T \geqslant 0)$$

is valid; this estimate becomes a strict equality only on a family of controls $mu^\circ(w, v)$ $(0 \leqslant m \leqslant 1)$ [6].

8.1.2. The right derivative $T^{\circ\cdot}(w, u, v^\circ(w)) > -1$ for $w \in D_1{}^\circ \cap [r(w) > 0]$.

Proof. Elementary calculations allow us to obtain the equality

$$T^{\circ\cdot} = -1 + x^2 (p_2 (p_2 - y_\alpha) / x^2 + p_2^{\cdot} / | x | + 1) / l_1^2 (w, p_2) \qquad (8.1)$$

where the derivative $p_2^{\cdot}$ is computed from the equation

$$p_2^{\cdot} q^{(1)} (w, p_2) = - | x | q^{(1)} (w, p_2) - (p_2 - y_\alpha) | x | / l_1 - p_2 y_\beta^2 / | x | l_2 - p_2 y_\alpha / l_1^2 - x^2 / l_1^2 + P_2 (w, u) + P_3 (w, v^\circ (w)) \qquad (8.2)$$

$$P_2 (w, u) = + | u | + (p_2 - y_\alpha) u_\alpha / l_2 + | y_\beta | u_\beta / l_2$$

$$P_3 (w, v^\circ (w)) = (p_2 - y_\alpha) v_\alpha^\circ / l_2 + | y_\beta | v_\beta^\circ / l_2 = -1$$

$$l_1 = l_1 (w, p_2), \quad l_2 = l_2 (w, p_2), \quad p_2 = p_2 (w)$$

The estimate $q^{(1)} (w, p) > 0$ is valid for $r (w) > 0$; therefore, after substituting $p_2^{\cdot}$ from Eq. (8.2) into Eq. (8.1) we obtain

$$T^{\circ\cdot} = -1 + l_1^{-2} (q^{(1)})^{-1} (l_1 l_2 [- p_2 / l_1 - (p_2 - y_\alpha) / l_2] + | x | \cdot P_2 (w, u)) \qquad (8.3)$$

The estimate $- p_2 / l_1 - (p_2 - y_\alpha) / l_2 > | x | / l_1^2 > 0$ is a consequence of the estimate $q^{(1)} (w, p_2) > 0$, while the estimate $P_2 (w, u) \geqslant 0$ is obvious. As a result, formula (8.3) completes the proof of assertion 8.1.2.

8.1.3. As was noted at the end of the proof of Lemma 7.1, when $w (0) \in D_1^\circ \cap [r (w) = 0]$ the inclusion $w^{(1)} (t) \in D_1^\circ \cap [r (w) = 0]$ is preserved only on trajectory $w_1^{(1)}$, while the remaining trajectories generated by the pairs $u (w, v)$, $v^\circ (w)$ either repeat the trajectory $w_1^{(1)}$ to within a set of measure zero or take the position into set W_0. These arguments establish the estimate $T^\circ [w] \leqslant T [u, v^\circ (w)]$ for $w \in D_1^\circ \cap [r (w) = 0]$.

8.1.4. The obvious equality $T [u^\circ, v^\circ] = T [u^\circ, v]$ and the continuity of the functions $p_2 (w)$, $T^\circ (w)$ on the set $D_1^\circ \cup M$ [6] complete the proof of Theorem 8.1.

The geometric interpretation of the optimal motion repeats the interpretation in Sect. 6 with the difference that the circle in Fig. 2 always has the center at the point (1, 0) and that the quantity $1 - s (w)$ is plotted along the horizontal axis instead of the quantity $| x | / y_\alpha^2$.

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